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Explicit Inertial Range Renormalization Theory in a Model for Turbulent Diffusion

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The inertial range for a statistical turbulent velocity field consists of those scales that are larger than the dissipation scale but smaller than the integral scale. Here the complete scale-invariant explicit inertial range renormalization theory for all the higher-order statistics of a diffusing passive scalar is developed in a model which, despite its simplicity, involves turbulent diffusion by statistical velocity fields with arbitrarily many scales, infrared divergence, long-range spatial correlations, and rapid fluctuations in time—such velocity fields retain several characteristic features of those in fully developed turbulence. The main tool in the development of this explicit renormalization theory for the model is an exact quantum mechanical analogy which relates higher-order statistics of the diffusing scalar to the properties of solutions of a family of N-body parabolic quantum problems. The canonical inertial range renormalized statistical fixed point is developed explicitly here as a function of the velocity spectral parameter ε, which measures the strength of the infrared divergence: for $\varepsilon < 2$, mean-field behavior in the inertial range occurs with Gaussian statistical behavior for the scalar and standard diffusive scaling laws; for $\varepsilon > 2$ a phase transition occurs to a fixed point with anomalous inertial range scaling laws and a non-Gaussian renormalized statistical fixed point. Several explicit connections between the renormalization theory in the model and intermediate asymptotics are developed explicitly as well as links between anomalous turbulent decay and explicit spectral properties of Schrödinger operators. The differences between this inertial range renormalization theory and the earlier theories for large-scale eddy diffusivity developed by Avellaneda and the author in such models are also discussed here.

KEY WORDS: Turbulent diffusion; inertial range renormalization; long-range correlations; quantum mechanical analogy.

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1. INTRODUCTION

The equation for a passive scalar $T(x, t)$ diffusing in a random incompressible velocity field is given by

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = \kappa \Delta T$$
$$T|_{t=0} = T_0(x) \tag{1}$$

In (1), v is a prescribed incompressible velocity field, i.e., $\operatorname{div} v = 0$, and $\kappa > 0$ is the coefficient of molecular diffusion. In fully developed turbulence[1-3] the velocity field is statistical with arbitrarily many spatial scales and long-range spatial correlations while fluctuating rapidly in time. The inertial range for the velocity field consists of those scales that are larger than the dissipation scale but smaller than the integral scale; an important problem in turbulence theory is to determine the inertial range renormalization theory[3-5] for the passive scalar in (1) at arbitrarily high Reynolds number in order to establish the universal features of passive scalar dynamics in the inertial range. This problem is of practical interest in its own right; it also serves as an important prototype problem for turbulence theories involving the Navier–Stokes equations, since the equation in (1) is statistically nonlinear even though this equation is linear for a given realization. One major difficulty in developing the inertial range renormalization theory for (1) at arbitrarily large Reynolds numbers is the fact that the velocity fields in fully-developed turbulence exhibit strong infrared divergences in the high-Reynolds-number limit.[3-5]

The goal in this paper is to develop the complete explicit inertial range renormalization theory in a model[6-8] for (1) which, despite its simplicity, nevertheless involves statistical velocity fields with arbitrarily many scales, infrared divergence, long-range spatial correlations, and rapid fluctuations in time. These simplified models are the special case of (1) given by

$$\frac{\partial T}{\partial t} + v(x, t)\frac{\partial T}{\partial y} = \kappa \Delta T$$
$$T|_{t=0} = T_0(x, y) \tag{2}$$

where the concentration $T(t, x, y)$ depends on the two spatial variables x, y. In (2) the velocity field is a stationary, zero-mean, Gaussian random field with correlation function given by

$$\langle v(x + x', t + t')\, v(x', t')\rangle = \tilde{R}(|t|)\, R_\delta^\varepsilon(x) \tag{3}$$

where

$$R_\delta^\varepsilon(x) = \int e^{2\pi ixk} |k|^{1-\varepsilon} \rho_\infty(|k|) \rho_0\left(\frac{|k|}{\delta}\right) dk \tag{4}$$

and ε is a parameter with $-\infty < \varepsilon < 4$. It is worth mentioning here that this choice of the parameter ε is 2/3 of that from refs. 4 and 5. The model in (2) has been nondimensionalized with dissipation scales of unit length so that $\rho_\infty(|k|)$, a fixed, rapidly decreasing function with $\rho_\infty(0) \equiv 1$, represents the dissipative decay of the velocity field. In (4), the integral scale has been nondimensionalized to correspond to wave numbers $|k| = O(\delta)$; thus, the function $\rho_0(|k|)$ is a fixed infrared cutoff: $\rho_0(|k|)$ is a smooth positive function with $\rho_0(|k|) \equiv 1$ for $|k| > 2$ and $\rho_0(|k|) \equiv 0$ for $|k| < 1$. The inertial range in the model at arbitrarily high Reynolds numbers corresponds to the wave numbers k with

$$\delta \ll |k| \ll 1 \tag{5}$$

and $\delta \to 0$ in the high-Reynolds-number limit.[6] For example, in standard Kolmogoroff turbulence theory,[3,6] $\delta = (Re)^{-3/4}$ with Re the Reynolds number, and $\varepsilon = 8/3$ in (4). There is strong infrared divergence in the velocity field in (3), (4) in the high-Reynolds-number limit for ε with $2 < \varepsilon < 4$ because the mean energy diverges, i.e.,

$$\langle v^2(x, t)\rangle \to \infty \qquad \text{as} \quad \delta \to 0 \qquad \text{for} \quad 2 < \varepsilon < 4 \tag{6}$$

In this paper, the additional simplifying assumption is made that the velocity statistics are Gaussian white noise in time so that

$$\tilde{R}(|t|) = V^2 \delta(t) \tag{7}$$

with $\delta(t)$ the Dirac delta function and V a constant. Next, I briefly summarize and discuss some of the features for the inertial range renormalization theory for (2) which are developed in the remainder of this paper.

With suitable zero-mean Gaussian random initial data $T_0(x, y)$ for (2) [see (27), (28) below], the complete set of statistical quantities for $T(t, x, y)$ at any later time involves the vector $\mathbf{P}$ of higher-order correlation functions defined by $\mathbf{P} = (P_{2N})$, $N = 1, 2, 3, 4, \ldots$, with

$$P_{2N}(t, \mathbf{x}, \mathbf{y}) = \left\langle \prod_{i=1}^{2N} T(t, x_i, y_i) \right\rangle \tag{8}$$

and $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i)$, $\mathbf{x}, \mathbf{y} \in R^{2N}$; with Gaussian random initial data, all odd statistics P_{2N-1} automatically vanish at any later time. The first crucial step in the renormalization theory is developed in Section 2; there it is established that despite the infinities caused by the infrared divergence in (6), the vector of statistical correlations **P** defined in (8) remains finite in the high-Reynolds-number limit as $\delta \to 0$ for $-\infty < \varepsilon < 4$ with a *structure independent of the infrared cutoff* ρ_0. This is achieved in Section 2 through an exact quantum mechanical analogy for (2) under the assumption in (7) developed through suitable manipulations of function space integrals and the Feynman–Kac formula[9]; this parabolic quantum mechanical analogy yields representation formulas for $P_{2N}(t, \mathbf{x}, \mathbf{y})$ via suitable solutions of quantum N-body problems and results in an infinite family of explicit Fokker–Planck equations for each of the higher-order statistics P_{2N}. This is the main technical tool exploited throughout this paper.

With the exact quantum mechanical analogy from Section 2, the explicit inertial range renormalization theory is developed in Sections 3 and 4. The strategy for inertial range renormalization involves finding self-consistent scaling laws $(\alpha(\lambda), \beta(\lambda)) = \Lambda(\lambda)$ and a corresponding amplitude scaling $A(\lambda)$ so that with the transformation

$$\begin{aligned} x' &= \lambda x \\ y' &= \alpha(\lambda)\, y \\ t' &= \beta(\lambda)\, t \end{aligned} \tag{9}$$

all the normalized correlation functions $R^{\Lambda}(A^{-1}\mathbf{P})$ defined componentwise by

$$(R^{\Lambda}(A^{-1}\mathbf{P}))_{2N} = \left(A(\lambda)^{-N} P_{2N}\left(\frac{t'}{\beta(\lambda)}, \frac{x'}{\lambda}, \frac{y'}{\alpha(\lambda)}\right)\right) \tag{10}$$

approach a fixed point in the inertial range scaling limit as $\lambda \to 0$. In (10) the notation $A^{-1}\mathbf{P}$ denotes the amplitude scaling of correlation functions with $(A^{-1}\mathbf{P})_N = A(\lambda)^{-N} P_{2N}$, while R^{Λ} denotes the rescaling transformation

$$(R^{\Lambda}\mathbf{P})_N = P_{2N}\left(\frac{t}{\beta(\lambda)}, \frac{x}{\lambda}, \frac{y}{\alpha(\lambda)}\right)$$

I remind the reader that with (5), the inertial range scaling theories should necessarily apply in the limit with $\lambda \to 0$ after the high-Reynolds-number limit $\delta \to 0$ has been achieved (see Section 5 for further discussion). In particular, in Section 4, the existence of a canonical renormalized fixed point for the statistics is established in this limit, i.e., there exists $\mathbf{P}_\varepsilon$ satisfying the limit in (10) as $\lambda \to 0$ together with the scale-invariant fixed-point equation

$$R^{\Lambda}\mathbf{P}_\varepsilon = \mathbf{P}_\varepsilon \tag{11}$$

The fixed-point vector of statistics $\mathbf{P}_\varepsilon$ is universal in the sense that it is independent of the infrared cutoff ρ_0 in the velocity and depends only on the velocity spectrum as characterized by the parameter ε. However, both the scaling laws $\alpha(\lambda)$, $\beta(\lambda)$, $A(\lambda)$ in (9) and the structure of the fixed point $\mathbf{P}_\varepsilon$ depend strongly on ε—there is a phase transition[6–8, 10] from Gaussian mean-field behavior for $\varepsilon < 2$ to non-Gaussian anomalous behavior for $\varepsilon > 2$. For the mean-field regime with $\varepsilon < 2$, the canonical inertial range scaling laws in (11) are the ordinary diffusive scalings,

$$\begin{aligned} \alpha(\lambda) &= \lambda \\ \beta(\lambda) &= \lambda^2 \end{aligned} \tag{12}$$

and:

- The canonical fixed point $\mathbf{P}_\varepsilon$ is Gaussian for $\varepsilon < 2$ with dependence on the viscous decay factor $\rho_\infty(|k|)$.

For the regime $2 < \varepsilon < 4$ with infrared divergence and long-range spatial correlations in the velocity field, the canonical scaling laws in (12) are superdiffusive with dependence on ε and given by

$$\begin{aligned} \alpha(\lambda) &= \lambda^{\varepsilon/2} \\ \beta(\lambda) &= \lambda^2 \end{aligned} \tag{13}$$

and:

- The canonical fixed point $\mathbf{P}_\varepsilon$ is non-Gaussian for $2 < \varepsilon < 4$ provided $\kappa \neq 0$; for this regime, the fixed point is independent of both cutoffs, $\rho_0(|k|)$ and $\rho_\infty(|k|)$, as expected in an inertial range scaling theory for turbulence.

Section 3 contains a complete discussion of the inertial range renormalization of the second-order statistics P_2, including explicit formulas exhibiting a phase transition in ε to anomalous turbulent decay for $2 < \varepsilon < 4$ from the usual diffusive decay for $\varepsilon < 2$; in the quantum mechanical analogy for the second-order statistics, the phase transition in the inertial range scaling theory at $\varepsilon = 2$ manifests itself as a transition from the behavior of solutions of the free-space Schrödinger equation for $\varepsilon < 2$ to the behavior of a Schrödinger equation with only pure point spectrum for $\varepsilon > 2$. With the material in Section 3 as motivation, the complete renormalization theory for **P** is developed in Section 4; there is a nice explicit connection between this inertial range renormalization theory and intermediate asymptotics, as suggested in earlier work,[11,12] and this is also discussed in Section 4. Section 5 includes a brief discussion of the similarities and differences with the large-scale eddy diffusivity renormalization theory for (2) developed earlier[6–8] by Avellaneda and the author as well as other recent work[13] in the literature for the model in (2). For pedagogical reasons, the results in this paper are presented largely as formal calculations; however, these calculations can be made mathematically rigorous at the expense of obscuring the main ideas, so these technical considerations are omitted here. It is worth mentioning here that there are other interesting analogies between various facets of turbulence theory and quantum theory which have been developed recently.[14]

2. AN EXACT QUANTUM MECHANICAL ANALOGY AND RENORMALIZATION IN THE HIGH-REYNOLDS-NUMBER LIMIT

To develop the exact quantum analogy, first I recall the representation formula for a general solution of a parabolic quantum mechanical problem in N dimensions via function space integrals through the Feynman–Kac formula[9]: The function $\psi(t, x)$ with $x \in R^N$ satisfies the N-dimensional parabolic quantum mechanics problem

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \kappa \Delta_N \psi + V(x, t)\psi \\ \psi\,|_{t=0} &= \psi_0(x) \end{aligned} \tag{14}$$

if and only if

$$\psi(t, x) = E_{\boldsymbol{\beta}}\left[\left\{\exp\left[\int_0^t V(x + (2\kappa)^{1/2}\,\boldsymbol{\beta}(s), t - s)\, ds\right]\right\} \times \psi_0(x + (2\kappa)^{1/2}\,\boldsymbol{\beta}(t))\right] \tag{15}$$

In (15), $\boldsymbol{\beta}(s)$ denotes a realization of N-dimensional Brownian motion with $\boldsymbol{\beta}(0) = 0$ and $E_{\boldsymbol{\beta}}[\cdot]$ denotes the function space integral obtained by averaging over $\boldsymbol{\beta}$. I utilize the equivalence between (14) and (15) at several points below in developing the exact quantum analogy for higher statistics.

Next, following earlier work of Avellaneda and the author,[6–8] I use (14) and (15) to represent the solution of (2) as a function space integral. With the Fourier representation

$$T_0(x, y) = \iint e^{2\pi i(x\eta + yk)}\, \hat{T}_0(\eta, k)\, d\eta\, dk \tag{16}$$

the solution of (2) is given by

$$T(x, y, t) = \iint e^{2\pi i y k} \psi(t, x, k, \eta)\, \hat{T}_0(\eta, k)\, d\eta\, dk \tag{17}$$

where $\psi(t, x, k, \eta)$ solves the parabolic (imaginary-time) quantum problem

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \kappa \psi_{xx} - [2\pi i k v(x, t) + \kappa 4\pi^2 k^2]\psi \\ \psi|_{t=0} &= e^{2\pi i x \eta} \end{aligned} \tag{18}$$

Thus, according to (14) and (15), the solution of (2) through (18) is given by the function space integral representation,

$$\begin{aligned} \psi(t, x, k, \eta) = {} & [\exp(-\kappa 4\pi^2 k^2 t)] \\ & \times E_\beta\left[\left\{\exp\left[-2\pi i k \int_0^t v(x + (2\kappa)^{1/2}\beta(s), t - s)\, ds\right]\right\}\right. \\ & \left.\times \exp\{2\pi i[x + (2\kappa)^{1/2}\beta(s)]\eta\}\right] \end{aligned} \tag{19}$$

for a fixed realization of the velocity field $v(x, t)$ with statistics in (3), (4). Here and below $\langle\cdot\rangle_v$ denotes ensemble averaging over the velocity statistics. With (16)–(19) one calculates the formula

$$\begin{aligned} & \prod_{i=1}^{N} T(t, x_i, y_i) \\ & \quad = \iint_{R^N \times R^N} [\exp(2\pi i \mathbf{y}\cdot\mathbf{k}) \exp(-\kappa 4\pi^2 |\mathbf{k}|^2 t)] \prod_{j=1}^{N} \hat{T}_0(\eta_j, k_j) \\ & \quad \times E_{\boldsymbol\beta}\left[\exp\left(2\pi i \sum_{j=1}^{N} \left\{-k_j \int_0^t v(x_j + (2\kappa)^{1/2}\beta_j(s), t - s)\, ds\right.\right.\right. \\ & \quad \left.\left.\left. + [x_j + (2\kappa)^{1/2}\beta_j(s)]\,\eta_j\right\}\right)\right] d\eta\, dk \end{aligned} \tag{20}$$

with $\mathbf{y} = (y_1, \ldots, y_N)$. To compute the average over the velocity statistics, $\langle \prod_{i=1}^{N} T(t, x_i, y_i) \rangle_v$, it follows from the formula in (20) that one only needs to calculate the average

$$E_{\boldsymbol{\beta}}\left[\left\{\exp\left[2\pi i \sum_{j=1}^{N} (x_j + (2\kappa)^{1/2} \beta_j(s))\, \eta_j\right]\right\} \times \left\langle \exp\left[-2\pi i \sum_{j=1}^{N} k_j \int_0^t v(x_j + (2\kappa)^{1/2} \beta_j(s), t-s)\, ds\right]\right\rangle_v\right] \tag{21}$$

For each fixed realization of Brownian motion $\boldsymbol{\beta}(s)$, the average over the velocity statistics involves the characteristic function of a zero-mean Gaussian random variable and thus this average can be calculated directly.[6–8] For velocity statistics with the form in (3), (4), the result is that the expression in (21) is given by

$$E_{\boldsymbol{\beta}}\left[\left\{\exp\left[2\pi i \sum_{j=1}^{N} (x_j + (2\kappa)^{1/2} \beta_j(s))\, \eta_j\right]\right\} \times \exp\left(\frac{-4\pi^2}{2} \sum_{i,j=1}^{N} k_i k_j \int_0^t \int_0^t R_\delta^\varepsilon(x_i - x_j + (2\kappa)^{1/2} [\beta_i(s) - \beta_j(s')]) \times \tilde{R}(|s - s'|)\, ds\, ds'\right)\right] \tag{22}$$

For the special case assumed in this paper of Gaussian white noise velocity statistics in time, with (7) the formula in (22) reduces to

$$E_{\boldsymbol{\beta}}\left[\left\{\exp\left[2\pi i \sum_{j=1}^{N} (x_j + (2\kappa)^{1/2} \beta_j(s))\, \eta_j\right]\right\} \times \exp\left\{-4\pi^2 V^2 \sum_{i,j=1}^{N} k_i k_j \int_0^t R_\delta^\varepsilon(x_i - x_j + (2\kappa)^{1/2} [\beta_i(s) - \beta_j(s)])\, ds\right\}\right] \tag{23}$$

Next, one recognizes that the expression in (23) is nothing else but the Feynman–Kac formula in (15) for an N-dimensional parabolic quantum mechanics problem with potential

$$V_N(\mathbf{x}, \mathbf{k}) = -\sum_{i,j=1}^{N} 4\pi^2 V^2 k_i k_j R_\delta^\varepsilon(x_i - x_j) \tag{24}$$

Thus, through (14), the function space integral in (23) is given exactly by $\psi_N(t, \mathbf{x}, \mathbf{k}, \boldsymbol{\eta})$, which is the solution of the parabolic quantum mechanics problem

$$\frac{\partial \psi_N}{\partial t} = \kappa \Delta_{\mathbf{x}} \psi_N + V_N(\mathbf{x}, \mathbf{k}) \psi_N \qquad (25)$$
$$\psi_N |_{t=0} = e^{2\pi i \mathbf{x} \cdot \boldsymbol{\eta}}$$

with the interaction potential given in (24) and $\Delta_{\mathbf{x}} = \sum_{i=1}^{N} \partial^2/\partial x_i^2$. By summarizing the calculations in (20)–(25), one obtains the following result:

Explicit Formula for Higher-Order Scalar Statistics:

$$\left\langle \prod_{i=1}^{N} T(t, x_i, y_i) \right\rangle_v = \iint_{R^N \times R^N} e^{2\pi i \mathbf{y} \cdot \mathbf{k}} e^{-\kappa 4\pi^2 |\mathbf{k}|^2 t} \psi_N(t, \mathbf{x}, \mathbf{k}, \boldsymbol{\eta}) \times \prod_{l=1}^{N} \hat{T}_0(\eta_l, k_l) \, d\eta \, dk \qquad (26)$$

where ψ_N solves the parabolic quantum mechanics problem in (25). This is the basic exact quantum mechanical analogy for the models in (2) which is utilized throughout this paper and elsewhere.[15,16]

The formula in (26) involves deterministic initial data. For Gaussian random initial data, the real-valued function $T_0(x, y)$ admits the spectral representation

$$T_0(x, y) = \iint e^{2\pi i(x\eta + yk)} \hat{T}_0(\eta, k) \, dW(\eta) \otimes dW(k) \qquad (27)$$

where

$$\hat{T}_0(-\eta, -k) = \overline{\hat{T}_0(\eta, k)}$$

and $dW(\eta) \otimes dW(k)$ is complex two-dimensional Gaussian white noise satisfying $\langle dW(\eta) \rangle$, $\langle dW(k) \rangle = 0$ and

$$\langle dW(\eta) \otimes dW(k), dW(\eta') \otimes dW(k') \rangle = \delta(\eta + \eta') \, \delta(k + k') \, d\eta \, d\eta' \, dk \, dk' \qquad (28)$$

If $\langle \cdot \rangle_0$ denotes averaging over Gaussian random initial data, then $\langle \cdot \rangle = \langle\langle \cdot \rangle_v \rangle_0$ denotes the average over both the random velocity statistics and the random initial data. It is a completely straightforward calculation to compute the higher-order scalar statistics $\langle \prod_{i=1}^{N} T(t, x_i, y_i) \rangle$ by averaging the formula in (26) over Gaussian random initial data by utilizing the formulas in (27), (28) in a cluster expansion, i.e., "Wick's Theorem." In this fashion, one obtains the following result:

Formula for Higher-Order Statistics with Gaussian Random Initial Data:

(A)

$$\left\langle \prod_{i=1}^{2N-1} T(t, x_i, y_i) \right\rangle = 0, \qquad N = 1, 2, 3,...$$

(B) For $N = 1, 2, 3,...$

$$\left\langle \prod_{i=1}^{2N} T(t, x_i, y_i) \right\rangle$$
$$= \sum_{\mathbf{i}^- \in \mathscr{P}} \iint_{R^N \times R^N} \{\exp[2\pi i \mathbf{y} \cdot (\pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k})] \exp(-8\pi^2 \kappa t\, |\mathbf{k}|^2)\}$$
$$\times \psi_{2N}(t, \mathbf{x}, \pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k}, \pi_{\mathbf{i}^-}\boldsymbol{\eta} - \pi_{\mathbf{i}^+}\boldsymbol{\eta}) \prod_{l=1}^{N} |\hat{T}_0(\eta_l, k_l)|^2 \, d\eta \, dk \qquad (29)$$

Here $\mathscr{P}$ denotes the set of all partitions of $2N$ numbers $\{1, 2,..., 2N\}$ into N pairs of integers $\{\{i_1^-, i_1^+\}, \{i_2^-, i_2^+\},..., \{i_N^-, i_N^+\}\} \in \mathscr{P}$, where by convention, $i_l^- < i_l^+$, $1 \leqslant l \leqslant N$, and $i_k^- < i_{k+1}^-$, $k = 1, 2,..., N-1$; the number of elements in $\mathscr{P}$ is $|\mathscr{P}| = (2N)!/2^N N!$. Given $\boldsymbol{\omega} \in R^N$, the vectors $\pi_{\mathbf{i}^-}(\boldsymbol{\omega})$, $\pi_{\mathbf{i}^+}(\boldsymbol{\omega}) \in R^{2N}$ are defined by the formulas

$$(\pi_{\mathbf{i}^-}(\boldsymbol{\omega}))_j = \begin{cases} \omega_l, & j = i_l^-, \quad 1 \leqslant l \leqslant N \\ 0, & \text{otherwise} \end{cases}$$
$$(\pi_{\mathbf{i}^+}(\boldsymbol{\omega}))_j = \begin{cases} \omega_l, & j = i_l^+, \quad 1 \leqslant l \leqslant N \\ 0, & \text{otherwise} \end{cases} \qquad (30)$$

2.1. Renormalization in the High-Reynolds-Number Limit

In formulas (26) and (29), the higher-order statistics for the scalar $T(t, x, y)$ have been expressed concisely through solutions of the family of parabolic quantum problems in (25) with the potentials defined in (24). In the high-Reynolds-number limit, $\delta \to 0$ and $R_\delta^\varepsilon(0) \nearrow \infty$ for $2 < \varepsilon < 4$ [see (5) and (6) above]; this infrared divergence in mean energy, in general, introduces strong divergences in the statistics for the scalar because the corresponding potentials in (24) obviously satisfy

$$V_N(\mathbf{x}, \mathbf{k}) \to -\infty \quad \text{for appropriate values of } \mathbf{k} \text{ as } \delta \to 0 \text{ for } 2 < \varepsilon < 4 \qquad (31)$$

The main fact established here in this subsection is that despite the infrared divergence implied by (31), nevertheless all the higher-order statistics for the scalar with Gaussian random initial data can be renormalized for all values of ε with $-\infty < \varepsilon < 4$. This is established for the model in (2)–(4) through one simple analytic fact together with some elementary algebraic identities.

The key analytic formula is the fact that $R_\delta^\varepsilon(0) - R_\delta^\varepsilon(x)$ is a well-behaved function in the limit $\delta \to 0$ for $2 < \varepsilon < 4$ despite the fact that $R_\delta^\varepsilon(0) \nearrow \infty$; the limit of $R_\delta^\varepsilon(0) - R_\delta^\varepsilon(x)$ is given by the formula

$$R^\varepsilon(0) - R^\varepsilon(x) = \int [1 - \cos(2\pi k x)]\, |k|^{1-\varepsilon} \rho_\infty(|k|)\, dk \qquad \text{for} \quad -\infty < \varepsilon < 4 \tag{32}$$

The limiting quantity in (32) is finite for any fixed x for $2 < \varepsilon < 4$ because $|1 - \cos(2\pi k x)| = O(|kx|^2)$ and this compensates for the divergence in $|k|^{1-\varepsilon}$ for $|k| \leqslant 1$ to produce an integrable function in k.

For Gaussian random initial data, it follows from the formula in (29) that one needs to assess the behavior of the potentials $V_{2N}(\mathbf{x}, \pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k})$ in the limit as $\delta \to 0$. Despite the divergence in (31) for certain values of $\mathbf{k}$, there is an elementary algebraic formula which permits one to rewrite $V_{2N}(\mathbf{x}, \tilde{\mathbf{k}})$ for the special vectors $\tilde{\mathbf{k}} = \pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k}$ solely through combinations of the convergent formula in (32). Without loss of generality, consider the case where $\pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k} \in R^{2N}$ has the form

$$\pi_{\mathbf{i}^-}\mathbf{k} - \pi_{\mathbf{i}^+}\mathbf{k} = (k_1, -k_1, k_2, -k_2, \ldots, k_N, -k_N) \tag{33}$$

[The more general case is reduced to this case through elementary transformations, as in (41)–(43) below.] With (24) one can verify readily the algebraic identity

$$\begin{aligned} &V_{2N}^{\varepsilon,\delta}(\mathbf{x}, (k_1, -k_1, \ldots, k_N, -k_N)) \\ &\quad = -8\pi^2 V^2 \left\{ \sum_{j=1}^{N} [R_\delta^\varepsilon(0) - R_\delta^\varepsilon(x_{2j} - x_{2j-1})]\, k_j^2 \right. \\ &\qquad \left. + \sum_{i<j} I_\delta^\varepsilon(x_{2i-1}, x_{2i}, x_{2j-1}, x_{2j})\, k_i k_j \right\} \end{aligned} \tag{34}$$

with the interaction potential $I_\delta^\varepsilon(y_1, y_2, y_3, y_4)$ given by

$$\begin{aligned} I_\delta^\varepsilon(y_1, y_2, y_3, y_4) = {} & [R_\delta^\varepsilon(y_1 - y_3) - R_\delta^\varepsilon(0)] + [R_\delta^\varepsilon(y_2 - y_4) - R_\delta^\varepsilon(0)] \\ & + [R_\delta^\varepsilon(0) - R_\delta^\varepsilon(y_1 - y_4)] + [R_\delta^\varepsilon(0) - R_\delta^\varepsilon(y_2 - y_3)] \end{aligned} \tag{35}$$

With (35), the algebraic formula in (34) and the analytic fact in (32) guarantee that for any $\mathbf{x}$,

$$V^{\varepsilon,\delta}_{2N}(\mathbf{x}, (k_1, -k_1,..., k_N, -k_N)) \text{ remains finite}$$
$$\text{in the high-Reynolds-number limit, } \delta \to 0 \tag{36}$$

I continue to utilize the formulas in (34), (35) in the limit $\delta \to 0$ in the remainder of this paper by simply substituting $R^\varepsilon(0) - R^\varepsilon(x)$ from (32) and retaining the same notation from (34), (35) with the superscript δ deleted. Since, from (29), the higher-order statistics for the scalar in the high-Reynolds-number limit are expressed through the parabolic quantum problems in (25) with the finite renormalized potentials in (34), (35) with $\delta = 0$, this completes the discussion of the high-Reynolds-number renormalization. Of course, the function $R^\varepsilon(0) - R^\varepsilon(x)$ has a completely different large $|x|$ asymptotic behavior for ε with $2 < \varepsilon < 4$ than for $\varepsilon < 2$ and this results in a radically different behavior for solutions of the quantum problem in (25) in these two regimes. This manifests itself in the completely different inertial range renormalization theory in these two regimes as described earlier in (9)–(11) of the introduction and developed in detail in Sections 3 and 4 below.

2.2. Fokker–Planck Equations for the Higher-Order Statistics in the High-Reynolds-Number Limit

Consider the function $\Phi_{2N}(t, \mathbf{x}, \mathbf{y})$ with $\mathbf{x} \in R^{2N}$ and $\mathbf{y} \in R^N$ defined by

$$\Phi_{2N}(t, \mathbf{x}, \mathbf{y}) = \iint_{R^N \times R^N} e^{2\pi i \mathbf{y}\cdot\mathbf{k}} e^{-8\pi^2 \kappa t |\mathbf{k}|^2} \psi_{2N}(t, \mathbf{x}, (k_1, -k_1,..., k_N, -k_N))$$
$$\times \prod_{l=1}^{N} |\hat{T}_0(\eta_l, k_l)|^2 \, d\eta \, dk \tag{37}$$

where ψ_{2N} solves the parabolic quantum problem in (25) with the renormalized potential $V^\varepsilon_{2N}(\mathbf{x}, (k_1, -k_1,..., k_N, -k_N))$ defined in (34) and (35) in the high-Reynolds-number limit. With (25), (34), (35), and (37), it follows immediately by a direct calculation that $\Phi_{2N}(t, \mathbf{x}, \mathbf{y})$ satisfies the renormalized *Fokker–Planck equation*,

$$\frac{\partial \Phi_{2N}}{\partial t} = \kappa \Delta_{\mathbf{x}} \Phi_{2N} + 2\kappa \Delta_{\mathbf{y}} \Phi_{2N} + 2V^2 \sum_{j=1}^{N} [R^\varepsilon(0) - R^\varepsilon(x_{2j} - x_{2j-1})] \frac{\partial^2}{\partial y_j^2} \Phi_{2N}$$
$$+ V^2 \sum_{i \neq j} I^\varepsilon(x_{2i-1}, x_{2i}, x_{2j-1}, x_{2j}) \frac{\partial^2}{\partial y_i \, \partial y_j} \Phi_{2N} \tag{38}$$

with initial data

$$\Phi_{2N}\,|_{t=0}=\prod_{i=1}^{N} R_0(x_{2i-1}-x_{2i},\, y_i) \tag{39}$$

Here

$$\Delta_{\mathbf{y}}=\sum_{i=1}^{2N}\frac{\partial^2}{\partial y_i^2},\qquad \Delta_{\mathbf{x}}=\sum_{i=1}^{2N}\frac{\partial^2}{\partial x_i^2}$$

and $R_0(x, y)$ is the correlation function of the Gaussian random initial data; thus, R_0 is given by

$$R_0(x, y)=\langle T_0(x+x', y+y')\, T_0(x', y')\rangle_0 \tag{40}$$

With the solution of Eq. (38), the formula for the higher-order statistics in (29), renormalized in the high-Reynolds-number limit, is given concisely by

$$P_{2N}(t, \mathbf{x}, \mathbf{y})=\left\langle \prod_{i=1}^{2N} T(t, x_i, y_i)\right\rangle=\sum_{\mathbf{i}\in\mathscr{P}} \Phi_{2N}(t, V_{\mathbf{i}}\mathbf{x}, W_{\mathbf{i}}\mathbf{y}) \tag{41}$$

where $V_{\mathbf{i}^-}: R^{2N}\to R^{2N}$ and $W_{\mathbf{i}^-}: R^{2N}\to R^{N}$ are linear transformations defined as follows for each $\mathbf{i}^-\in\mathscr{P}$.

Let $\mathbf{e}_j\in R^{2N}$ be the standard orthonormal basis for R^{2N} with $1\leqslant j\leqslant 2N$; then

$$\begin{aligned} V_{\mathbf{i}^-}\mathbf{e}_{i_l^-}&=\mathbf{e}_{2l-1}, \qquad 1\leqslant l\leqslant N\\ V_{\mathbf{i}^-}\mathbf{e}_{i_l^+}&=\mathbf{e}_{2l}, \qquad 1\leqslant l\leqslant N \end{aligned} \tag{42}$$

and for any $\mathbf{y}\in R^{2N}$,

$$(W_{\mathbf{i}}(\mathbf{y}))_l=y_{i_l^-}-y_{i_l^+}, \qquad 1\leqslant l\leqslant N \tag{43}$$

3. INERTIAL RANGE RENORMALIZATION FOR THE SECOND-ORDER STATISTICS

In this section, the inertial range renormalization theory is developed for the second-order statistics

$$P_2(t, \mathbf{x}, \mathbf{y})=\langle T(t, x_1, y_1)\, T(t, x_2, y_2)\rangle \tag{44}$$

to illustrate the main features in detail of the general renormalization theory developed in Section 4. In this special case, it follows from (38), (40), (41) that in the high-Reynolds-number limit,

$$P_2(t, \mathbf{x}, \mathbf{y})=\Phi_2(t, x_1-x_2, y_1-y_2) \tag{45}$$

where $\Phi_2(t, x, y)$ satisfies the explicit Fokker–Planck equation

$$\frac{\partial \Phi_2}{\partial t} = 2\kappa \frac{\partial^2 \Phi_2}{\partial x^2} + 2\kappa \frac{\partial^2 \Phi_2}{\partial y^2} + 2V^2[R^\varepsilon(0) - R^\varepsilon(x)] \frac{\partial^2}{\partial y^2} \Phi_2 \tag{46}$$

with initial conditions

$$\Phi_2 |_{t=0} = R_0(x, y) \tag{47}$$

[There is a slight abuse of notation in (46), (47) above which should not confuse the reader.]

For the inertial range renormalization theory for the second-order statistics, one seeks self-consistent scaling laws as in (9) with parameters $(\alpha(\lambda), \beta(\lambda)) = \Lambda(\lambda)$ and $A(\lambda)$ so that the normalized second-order correlation functions $R^\Lambda A^{-1} P_2$ defined by

$$\begin{aligned} R^\Lambda A^{-1} P_2 &= [A(\lambda)]^{-1} P_2 \left(\frac{t}{\beta(\lambda)}, \frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\alpha(\lambda)} \right) \\ &= [A(\lambda)]^{-1} \Phi_2 \left(\frac{t}{\beta(\lambda)}, \frac{x_1 - x_2}{\lambda}, \frac{y_1 - y_2}{\alpha(\lambda)} \right) \end{aligned} \tag{48}$$

approach a nontrivial fixed point in the inertial range scaling limit as $\lambda \to 0$. With the notation motivated by (48) and (46), (47), $R^\Lambda A^{-1} \Phi_2 = \Phi_2^\lambda$ satisfies the rescaled Fokker–Planck equation

$$\frac{\partial \Phi_2^\lambda}{\partial t} = 2\kappa \frac{\lambda^2}{\beta} \frac{\partial^2}{\partial x^2} \Phi_2^\lambda + 2\kappa \frac{\alpha^2}{\beta} \frac{\partial^2}{\partial y^2} \Phi_2^\lambda + \frac{\alpha^2}{\beta} 2V^2 \left[R^\varepsilon(0) - R^\varepsilon\left(\frac{x}{\lambda}\right) \right] \frac{\partial^2}{\partial y^2} \Phi_2^\lambda \tag{49}$$

with the initial condition

$$\Phi_2^\lambda |_{t=0} = [A(\lambda)]^{-1} R_0 \left(\frac{x}{\lambda}, \frac{y}{\alpha(\lambda)} \right) \tag{50}$$

The scale-invariant solution $\Phi_{2,\varepsilon}$ that describes the inertial range renormalization depends crucially on the velocity statistics as manifested by the scaling behavior of the quantity

$$R^\varepsilon(0) - R^\varepsilon\left(\frac{x}{\lambda}\right) \quad \text{as} \quad \lambda \to 0 \tag{51}$$

The quantity in (51) exhibits a "phase transition" in its scaling behavior as the spectral parameter ε from (4) crosses from the region $\varepsilon < 2$ to the region $2 < \varepsilon < 4$. With $R^\varepsilon(0) - R^\varepsilon(x)$ defined in (32), it is a simple matter to use the Riemann–Lebesgue lemma because $|k|^{1-\varepsilon}\rho_\infty(|k|)$ is integrable for $\varepsilon<2$ and verify that

$$\lim_{\lambda\to 0}\left[R^\varepsilon(0)-R^\varepsilon\left(\frac{x}{\lambda}\right)\right]=R^\varepsilon(0) \qquad \text{for} \quad -\infty<\varepsilon<2 \tag{52}$$

On the other hand, for $2<\varepsilon<4$ it is a simple matter to rescale (32) utilizing the fact that $\rho_\infty(|k|)$ is smooth with $\rho_\infty(0)=1$ and compute the completely different scaling behavior

$$\lim_{\lambda\to 0}\left\{\left[R^\varepsilon(0)-R^\varepsilon\left(\frac{x}{\lambda}\right)\right]\lambda^{\varepsilon-2}\right\}=C_\varepsilon\,|x|^{\varepsilon-2} \qquad \text{for} \quad 2<\varepsilon<4 \tag{53}$$

with

$$C_\varepsilon=\int_{-\infty}^{\infty}[1-\cos(2\pi k)]\,|k|^{1-\varepsilon}\,dk \tag{54}$$

This different scaling behavior is a direct consequence of the long-range correlations in the velocity field for $\varepsilon>2$. I remind the reader that the integral in (54) is convergent for $2<\varepsilon<4$ and does not depend on the ultraviolet cutoff $\rho_\infty(|k|)$. On the other hand, the formula in (52) for $R^\varepsilon(0)$ for $\varepsilon<2$ depends on the ultraviolet cutoff $\rho_\infty(|k|)$. With (49)–(54), it is a simple matter to compute in a formal fashion the self-consistent scaling laws $A(\lambda)$ as well as the renormalized fixed point $\Phi_{2,\varepsilon}$ defining the second-order correlation functions in the inertial range. For simplicity in exposition, it is assumed below that the correlation function of the Gaussian random initial data in (40) is short range and satisfies

$$0<\overline{R_0}=\iint R_0(x,y)\,dx\,dy<+\infty \tag{55}$$

The Mean-Field Regime: $\epsilon<2$. First, consider the mean-field regime with $\varepsilon<2$. With the behavior in (52) and Eq. (49) for Φ_2^λ, the usual self-consistent diffusive scaling laws

$$\alpha(\lambda)=\lambda, \qquad \beta(\lambda)=\lambda^2, \qquad A(\lambda)=\lambda^2 \tag{56}$$

apply and guarantee, with (55), formally that $\Phi_2^\lambda\to\Phi_{2,\varepsilon}$ as $\lambda\to 0$, where $\Phi_{2,\varepsilon}$ satisfies the diffusion equation

$$\frac{\partial\Phi_{2,\varepsilon}}{\partial t}=2\kappa\frac{\partial^2}{\partial x^2}\Phi_{2,\varepsilon}+[2\kappa+2V^2R^\varepsilon(0)]\frac{\partial^2}{\partial y^2}\Phi_{2,\varepsilon} \tag{57}$$

with initial conditions

$$\Phi_{2,\varepsilon}|_{t=0} = \overline{R_0}\delta(x) \otimes \delta(y) \tag{58}$$

Of course, the solution $\Phi_{2,\varepsilon}$ can be written down explicitly as a multiple of the fundamental solution of the heat equation with enhanced y diffusivity, $2V^2R^\varepsilon(0)$, due to the random velocity fluctuations; this inertial range correlation function $\Phi_{2,\varepsilon}$ depends on the random initial data satisfying (55) through the constant $\overline{R_0}$. Clearly, Φ_2 is scale invariant with the scaling laws from (56), so that $R^A\Phi_{2,\varepsilon} = \Phi_{2,\varepsilon}$. The scale-invariant initial condition in (58) corresponds to Gaussian white-noise random initial data.

The Anomalous Scaling Regime: $2 < \epsilon < 4$. Next, consider the anomalous scaling regime with $2 < \varepsilon < 4$. I follow the same strategy as in the mean-field regime; however, as a consequence of the long-range velocity correlations for $\varepsilon > 2$, the inertial range scaling theory exhibits anomalous behavior. With the asymptotic scaling behavior in (53) for ε with $2 < \varepsilon < 4$, the equation for Φ_2^λ in (49) can be written in the form

$$\frac{\partial \Phi_2^\lambda}{\partial t} = 2\kappa \frac{\lambda^2}{\beta} \frac{\partial^2}{\partial x^2} \Phi_2^\lambda + 2\kappa \frac{\alpha^2}{\beta} \frac{\partial^2}{\partial y^2} \Phi_2^\lambda + \frac{\alpha^2 \lambda^{2-\varepsilon}}{\beta} 2V^2 \left\{ \left[R^\varepsilon(0) - R^\varepsilon\left(\frac{x}{\lambda}\right) \right] \lambda^{\varepsilon-2} \right\} \frac{\partial^2}{\partial y^2} \Phi_2^\lambda \tag{59}$$

According to (53), the expression in braces in (59) is independent of λ and behaves like $C_\varepsilon |x|^{\varepsilon-2}$ in the inertial range scaling regime with $\lambda \to 0$; thus, unlike the mean-field regime, for $2 < \varepsilon < 4$, there are three independent scaling coefficients in (59). The simple mean-field scaling from (56) implies the coefficient $\alpha^2\lambda^{2-\varepsilon}/\beta \to \infty$ and the second-order correlations are not renormalized in this regime. The alternative needed to guarantee a finite nontrivial renormalized equation as $\lambda \to 0$ in (59) is to choose

$$\frac{\lambda^2}{\beta} = 1, \qquad \frac{\alpha^2 \lambda^{2-\varepsilon}}{\beta} = 1$$

With this choice, it follows that $\alpha^2/\beta \to 0$ and

$$\beta = \lambda^2, \qquad \alpha = \lambda^{\varepsilon/2}, \qquad A(\lambda) = \lambda^{1+\varepsilon/2} \tag{60}$$

With the scaling laws in (60), it follows from (59), (55), and (53) that formally Φ_2^λ converges as $\lambda \to 0$ to $\Phi_{2,\varepsilon}$, where $\Phi_{2,\varepsilon}$ satisfies the fixed-point equation

$$\frac{\partial \Phi_{2,\varepsilon}}{\partial t} = 2\kappa \frac{\partial^2}{\partial x^2} \Phi_{2,\varepsilon} + 2V^2 C_\varepsilon |x|^{\varepsilon-2} \frac{\partial^2}{\partial y^2} \Phi_{2,\varepsilon} \qquad \text{for} \quad 2 < \varepsilon < 4 \tag{61}$$

with initial condition

$$\Phi_{2,\varepsilon}\,|_{t=0} = \overline{R_0}\,\delta(x) \otimes \delta(y) \tag{62}$$

The solution $\Phi_{2,\varepsilon}$, which defines the renormalized second-order correlations, is a multiple of the fundamental solution of Eq. (61) and obviously is completely independent of the ultraviolet cutoff $\rho_\infty(|k|)$; unlike the mean-field regime, Eq. (61) exhibits increasingly strong diffusion as $|x| \to \infty$ as a consequence of the infrared divergence of mean energy [see (6) above] and the long-range correlations in the velocity field as manifested in (53) for $2 < \varepsilon < 4$—such strong enhanced diffusion is typical in fully developed turbulence.[2,3] The Green's function defined in (61) and (62) has a completely different character than the standard fundamental solution for a heat equation from (59) corresponding to the mean-field behavior for $\varepsilon < 2$ (see Section 3.1 below). The reader can verify readily that with the scaling laws in (60), $R^A \Phi_{2,\varepsilon} = \Phi_{2,\varepsilon}$ as should necessarily be satisfied at the renormalized fixed point. This completes the formal renormalization theory for the second-order statistics.

3.1. Anomalous Turbulent Decay for $2 < \varepsilon < 4$

Here I illustrate the rather different behavior of the statistical energy decay in the inertial range for the anomalous regime $2 < \varepsilon < 4$ compared with the ordinary diffusive decay of energy in the mean-field regime with $\varepsilon < 2$. This is achieved through the quantum mechanical analogy developed in Section 2; here I utilize the fact that the parabolic quantum problems associated with the inertial range fixed point in (61) for $2 < \varepsilon < 4$ have a completely different spectral character than the standard behavior associated with Eq. (57), which is valid for $\varepsilon < 2$.

To demonstrate these facets of anomalous energy decay in a simple fashion, consider Gaussian random initial data $T_0(y)$, which is a function of y alone, so that the correlation function $R_0(y) = \langle T_0(y + y')\, T_0(y') \rangle$ satisfies

$$0 < \overline{R_0} = \int R_0(y)\, dy < +\infty \tag{63}$$

With the choice $A(\lambda) = \lambda$ for $\varepsilon < 2$ and $A(\lambda) = \lambda^{\varepsilon/2}$ for $2 < \varepsilon < 4$, the same respective scaling laws in (56) and (60) for $\alpha(\lambda)$, $\beta(\lambda)$ yield the same inertial range limit equations in (57) and (61) for the two regions as developed earlier; however, in this case with $T_0(y)$, the inertial range scale-invariant initial data is given by

$$\Phi_{2,\varepsilon}\,|_{t=0} = \overline{R_0}(1 \otimes \delta(y)) \tag{64}$$

For $-\infty < \varepsilon < 2$, the standard Fourier representation for the solution of (57) with the initial data in (64) is given by

$$\Phi_{2,\varepsilon}(t, y) = \overline{R_0} \int_{-\infty}^{\infty} e^{2\pi i k y} e^{-(2\kappa + 2V^2 R^\varepsilon(0))\, t 4\pi^2 k^2}\, dk \tag{65}$$

By (44) and (45), the mean statistical energy in the scalar satisfies $\overline{[T_\varepsilon(t)]^2} = \Phi_{2,\varepsilon}(t, 0)$ and is given through (65) by

$$\begin{aligned} \overline{T_\varepsilon(t)^2} &= \overline{R_0} \int_{-\infty}^{\infty} e^{-(2\kappa + 2V^2 R^\varepsilon(0))\, t 4\pi^2 k^2}\, dk \\ &= \overline{R_0}\{4\pi[\kappa + V^2 R^\varepsilon(0)]\, t\}^{-1/2} \qquad \text{for} \quad -\infty < \varepsilon < 2 \end{aligned} \tag{66}$$

The formula in (66) illustrates that in the regime of spectral parameters with $-\infty < \varepsilon < 2$, the decay of mean energy in the inertial range scaling is the standard diffusive decay with a rate $t^{-1/2}$; the only role of the random fluctuations is to enhance this rate of decay through the coefficient $V^2 R^\varepsilon(0)$ in (66).

For the anomalous scaling regime with $2 < \varepsilon < 4$, more rapid decay of mean energy occurs in the inertial range due to long-range correlations in the velocity field. The solution of (61) with the initial data in (64) can be written in the form

$$\Phi_{2,\varepsilon}(t, x, y) = \int e^{2\pi i y \cdot k} \hat{\Phi}_{2,\varepsilon}(t, x, k)\, dk \tag{67}$$

where $\hat{\Phi}_{2,\varepsilon}(t, x, k)$ is the solution of the parabolic quantum problem

$$\frac{\partial}{\partial t} \hat{\Phi}_{2,\varepsilon} = 2\kappa \frac{\partial^2}{\partial x^2} \hat{\Phi}_{2,\varepsilon} - 8\pi^2 V^2 C_\varepsilon k^2 |x|^{\varepsilon - 2} \hat{\Phi}_{2,\varepsilon} \qquad \text{for} \quad 2 < \varepsilon < 4 \tag{68}$$

with

$$\hat{\Phi}_{2,\varepsilon}\big|_{t=0} = 1$$

The strong potential $|x|^{\varepsilon-2}$ for $2 < \varepsilon < 4$ guarantees that the Schrödinger operator on the right-hand side of (68) has only pure point spectrum[17] with a complete family of orthonormal eigenfunctions—this behavior contrasts strongly with the inertial range behavior for $\varepsilon < 2$ from (57) and (65), which involves only the continuous spectrum of a free-space Schrödinger operator. Let $\psi_j^0(x)$ be the L^2-normalized even eigenfunction with eigenvalue μ_j^0 for the normalized Schrödinger operator satisfying

$$\begin{gathered} \frac{d^2}{dx^2} \psi_j^0 - |x|^{\varepsilon-2} \psi_j^0 = -\mu_j^0 \psi_j^0 \\ \psi_j^0(-x) = \psi_j^0(x) \\ \int (\psi_j^0)^2\, dx = 1 \end{gathered} \tag{69}$$

with $0 < \mu_1^0 < \mu_2^0 < \mu_3^0 < \cdots$. It follows from simple rescalings of (69) in the eigenfunction expansion in (68) for $\hat{\Phi}_{2,\varepsilon}(t, x, k)$ that $\Phi_{2,\varepsilon}(t, x, y)$ is given for $t > 0$ by the formula

$$\Phi_{2,\varepsilon}(t, x, y)$$

$$= \int [\exp(2\pi i y k)] \left\{ \sum_{j=1}^{\infty} d_j^0 \exp[-\mu_j^0 2(4\pi^2)^{4/\varepsilon} \kappa^{1-2/\varepsilon} |V^2 C_\varepsilon|^{2/\varepsilon} |k|^{4/\varepsilon} t] \right.$$
$$\left. \times \psi_j^0((4\pi^2)^{2/\varepsilon} \kappa^{-1/\varepsilon} |V^2 C_\varepsilon|^{1/\varepsilon} |k|^{2/\varepsilon} x) \right\} dk \qquad (70)$$

where $d_j^0 = \int \psi_j^0(x)\, dx$. By (44) and (45), the mean statistical energy in the scalar in the inertial range satisfies $\overline{T_\varepsilon^2(t)} = \Phi_{2,\varepsilon}(t, 0, 0)$ and is given through (70) by

$$\overline{T_\varepsilon^2(t)} = \int \sum_{j=1}^{\infty} c_j^0 \exp[-\mu_j^0 2(4\pi^2)^{4/\varepsilon} \kappa^{1-2/\varepsilon} |V^2 C_\varepsilon|^{2/\varepsilon} |k|^{4/\varepsilon} t]\, dk$$
$$= t^{-\varepsilon/4} \kappa^{(2-\varepsilon)/4} |V^2 C_\varepsilon|^{-1/2} D_\varepsilon \qquad (71)$$

for $2 < \varepsilon < 4$, where D_ε is the finite constant

$$D_\varepsilon = \int \left(\sum_{j=1}^{\infty} c_j^0 e^{-2(4\pi^2)^{4/\varepsilon} \mu_j^0 |k|^{4/\varepsilon}} \right) dk$$

and $c_j^0 = \psi_j^0 \int \psi_j^0(x)\, dx$. The formula in (71) reveals more rapid turbulent decay of scalar energy in the inertial range at the rate $O(t^{-\varepsilon/4})$ in the anomalous scaling regime, $2 < \varepsilon < 4$, when compared with the ordinary viscous decay of mean energy in (66) for the mean-field regime with $-\infty < \varepsilon < 2$. The exponent of this decay rate for second-order correlations in the inertial range serves as an order parameter in the phase transition with the standard value 1/2 for $\varepsilon < 2$ and the anomalous value $\varepsilon/4$ for $2 < \varepsilon < 4$.

4. INERTIAL RANGE RENORMALIZATION FOR THE HIGHER-ORDER STATISTICS

Here the inertial range renormalization theory, as summarized in (9)–(13) of the introduction, is developed for the vector of higher-order statistics, $\mathbf{P} = (P_{2N})$, $N = 1, 2, 3, 4,\ldots$, with $P_{2N}(t, \mathbf{x}, \mathbf{y}) = \langle \prod_{i=1}^{2N} T(t, x_i, y_i) \rangle$. The overall strategy follows the one already developed in detail for the second-order statistics in Section 3 and utilizes the scaling behavior for the general Fokker–Planck equations in (38), (39) in the analogous fashion as in (56)–(62) of Section 3. It is assumed here that the correlation function of the Gaussian random initial data also satisfies the condition in (55) from Section 3. With the formula in (41) for $P_{2N}(t, \mathbf{x}, \mathbf{y})$ via a finite sum of terms involving $\Phi_{2N}(t, \mathbf{x}, \mathbf{y})$, it is clearly sufficient to develop the renormalization theory for the vector $\mathbf{\Phi} = (\Phi_{2N}(t, \mathbf{x}, \mathbf{y}))$, $N = 1, 2, 3,\ldots$. With the inertial range renormalization transformation $R^{A}A^{-1}\Phi$ defined componentwise by

$$(R^{A}A^{-1}\mathbf{\Phi})_N = [A(\lambda)]^{-N}\,\Phi_{2N}\left(\frac{t}{\beta(\lambda)}, \frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\alpha(\lambda)}\right) \tag{72}$$

for $N = 1, 2, 3,\ldots$, one seeks self-consistent scaling laws as in (9) with a nontrivial universal fixed point for $\mathbf{\Phi}$ in the limit, $\lambda \to 0$, which depends essentially only on the spectral parameter ε. With (38) and (39), it follows that $\Phi^{\lambda}_{2N} = (R^{A}A^{-1}\mathbf{\Phi})_N$ satisfies the Fokker–Planck equation,

$$\begin{aligned}\frac{\partial \Phi^{\lambda}_{2N}}{\partial t} &= \left(\frac{\lambda^2}{\beta}\right)\kappa\Delta_{\mathbf{x}}\Phi^{\lambda}_{2N} + \left(\frac{\alpha^2}{\beta}\right)2\kappa\,\Delta_{\mathbf{y}}\Phi^{\lambda}_{2N}\\ &\quad + \left(\frac{\alpha^2}{\beta}\right)2V^2\sum_{j=1}^{N}\left[R^{\varepsilon}(0) - R^{\varepsilon}\left(\frac{x_{2j}-x_{2j-1}}{\lambda}\right)\right]\frac{\partial^2}{\partial y_j^2}\Phi^{\lambda}_{2N}\\ &\quad + \left(\frac{\alpha^2}{\beta}\right)V^2\sum_{i\neq j}I^{\varepsilon}\left(\frac{x_{2i-1}}{\lambda}, \frac{x_{2i}}{\lambda}, \frac{x_{2j-1}}{\lambda}, \frac{x_{2j}}{\lambda}\right)\frac{\partial^2}{\partial y_i\,\partial y_j}\Phi^{\lambda}_{2N}\end{aligned} \tag{73}$$

with the initial data

$$\Phi^{\lambda}_{2N}\big|_{t=0} = \prod_{i=1}^{N}\,[A(\lambda)]^{-1}\,R_0\left(\frac{x_{2i-1}-x_{2i}}{\lambda}, \frac{y_i}{\alpha}\right) \tag{74}$$

The critical fact, in addition to (52)–(54), which is utilized in the renormalization theory for $\mathbf{\Phi}$ is the formula from (35), which establishes that the interaction potential $I^{\varepsilon}(y_1/\lambda, y_2/\lambda, y_3/\lambda, y_4/\lambda)$ is given by the sum over four terms involving the expression $R^{\varepsilon}(0) - R^{\varepsilon}((y_i - y_j)/\lambda)$; thus,

$$I^{\varepsilon}\left(\frac{x_{2i-1}}{\lambda}, \frac{x_{2i}}{\lambda}, \frac{x_{2j-1}}{\lambda}, \frac{x_{2j}}{\lambda}\right) \text{ has the same scaling}$$

properties in (52) and (53) as established earlier for $R^{\varepsilon}(0) - R^{\varepsilon}\left(\frac{x}{\lambda}\right)$

for the two regions $\varepsilon < 2$ and $2 < \varepsilon < 4$ (75)

The Mean-Field Regime, $\epsilon < 2$. With Eq. (73) for Φ^{λ}_{2N} and the behavior in (52), (75) for $\varepsilon < 2$, the same self-consistent diffusive scaling laws from (56) guarantee formally that $\Phi_{2N}^{\lambda} \to \Phi_{2N,\varepsilon}$, where $\Phi_{2N,\varepsilon}$ satisfies the diffusion equation

$$\frac{\partial \Phi_{2N,\varepsilon}}{\partial t} = \kappa\, \Delta_{\mathbf{x}} \Phi_{2N,\varepsilon} + [2\kappa + 2V^2 R^{\varepsilon}(0)]\, \Delta_{\mathbf{y}} \Phi_{2N,\varepsilon} \tag{76}$$

with initial conditions

$$\Phi_{2N,\varepsilon}\,|_{t=0} = \overline{R_0}^{N} \prod_{i=1}^{N} \delta(x_{2i-1} - x_{2i}) \otimes \delta(y_i) \tag{77}$$

It follows from (76), (77) that the solution of (76), $\Phi_{2N,\varepsilon}$, factorizes as a product involving second-order correlations, i.e.,

$$\Phi_{2N,\varepsilon} = \prod_{i=1}^{N} \Phi_{2,\varepsilon}(t, x_{2i-1} - x_{2i}, y_i) \tag{78}$$

with $\Phi_{2,\varepsilon}(t, x, y)$ given in (57). The factorization in (78) together with (41) guarantees that the renormalized statistics for the scalar in the mean-field regime, $\varepsilon < 2$, are Gaussian. It is a simple matter for the reader to verify that with the diffusive scaling law for $\alpha(\lambda)$, $\beta(\lambda)$ in (56), $R^{\Lambda}\mathbf{\Phi}_{\varepsilon} = \mathbf{\Phi}_{\varepsilon}$ for $\varepsilon < 2$, so that $\mathbf{\Phi}_{\varepsilon}$ determined by (76)–(78) is the inertial range renormalized fixed point.

The Anomalous Scaling Regime, $2 < \epsilon < 4$. With the important facts in (73), (75) and the asymptotic scaling behavior of $R(0) - R(x/\lambda)$ in (53), it follows by the same reasoning as utilized in Section 3 that with the same anomalous scaling laws in (60), $\Phi_{2N}^{\lambda} \to \Phi_{2N,\varepsilon}$ formally as $\lambda \to 0$. Here $\Phi_{2N,\varepsilon}$ satisfies the diffusion equation

$$\begin{aligned}\frac{\partial \Phi_{2N,\varepsilon}}{\partial t} &= \kappa\, \Delta_{\mathbf{x}} \Phi_{2N,\varepsilon} + 2V^2 C_{\varepsilon} \sum_{j=1}^{N} |x_{2j} - x_{2j-1}|^{\varepsilon-2} \frac{\partial^2}{\partial y_j^2} \Phi_{2N,\varepsilon} \\ &\quad + V^2 C_{\varepsilon} \sum_{i \neq j} \bar{I}^{\varepsilon}(x_{2i-1}, x_{2i}, x_{2j-1}, x_{2j}) \frac{\partial^2}{\partial y_i\, \partial y_j} \Phi_{2N,\varepsilon}\end{aligned} \tag{79}$$

with $\bar{I}^{\varepsilon}$ given by

$$\begin{aligned}\bar{I}^{\varepsilon}(y_1, y_2, y_3, y_4) &= |y_1 - y_4|^{\varepsilon-2} + |y_2 - y_3|^{\varepsilon-2} \\ &\quad - |y_1 - y_3|^{\varepsilon-2} - |y_2 - y_4|^{\varepsilon-2}\end{aligned} \tag{80}$$

and the initial conditions in (77), which are associated with Gaussian white noise random initial data. Furthermore, it is a simple matter for the reader to verify that with the anomalous scaling laws in (60) for $2 < \varepsilon < 4$ and

$\mathbf{\Phi}_\varepsilon = (\Phi_{2N,\varepsilon})$ defined through the solution of (79) and (77), $\mathbf{\Phi}_\varepsilon$ satisfies $R^A\mathbf{\Phi}_\varepsilon = \mathbf{\Phi}_\varepsilon$; thus, $\mathbf{\Phi}_\varepsilon$ is the inertial range fixed point for the higher-order statistics in the regime $2 < \varepsilon < 4$. In this anomalous regime, the statistics for the scalar do not factorize as in (75) and are non-Gaussian; in fact, under suitable hypotheses on the random initial data, the probability distribution function for the scalar at a single point is broader than Gaussian.(15,16) These last results complete the description of the inertial range renormalization theory for the scalar as summarized in (9)–(13) in the introduction.

4.1. Noncanonical Renormalized Fixed Points and Intermediate Asymptotics

In the preceding part of this section, it has been established that with the appropriate nonlinear inertial range scaling laws $\Lambda(\lambda) = (\alpha(\lambda), \beta(\lambda))$ and $A(\lambda)$ from (56) and (60) for $\varepsilon < 2$ and $2 < \varepsilon < 4$, respectively, the vector of correlation functions $\mathbf{P}$ from (8) converges as $\lambda \to 0$ to a renormalized fixed point $\mathbf{P}_\varepsilon$ satisfying $R^A\mathbf{P}_\varepsilon = \mathbf{P}_\varepsilon$. These fixed point correlation functions are determined via scale-invariant solutions $\mathbf{\Phi}_\varepsilon$ of the family of Fokker–Planck equations in (76) and (79) for $\varepsilon < 2$ and $2 < \varepsilon < 4$, respectively; these scale-invariant solutions satisfy $R^A\mathbf{\Phi}_\varepsilon = \mathbf{\Phi}_\varepsilon$.

Here, I point out that for each ε there are other nonlinear scaling laws, $\tilde{\Lambda}(\lambda) = (\tilde{\alpha}(\lambda), \tilde{\beta}(\lambda))$, and amplitude scaling $\tilde{A}(\lambda)$ so that a fixed point for the vector of correlation functions is achieved, i.e., $R^{\tilde{A}}(\tilde{\mathbf{\Phi}}_\varepsilon) = \tilde{\mathbf{\Phi}}_\varepsilon$, where $\tilde{\mathbf{\Phi}}_\varepsilon$ is determined by scale-invariant solutions for a family of Fokker–Planck equations. However, as discussed below, these *additional renormalized fixed points* $\tilde{\mathbf{\Phi}}_\varepsilon$ are *noncanonical* in the sense that they all arise from the universal fixed points determined by Eqs. (76) and (79) by simply rescaling these solutions $\mathbf{\Phi}_\varepsilon$ in appropriate inner and outer limits in the formal sense of intermediate asymptotics.(11) In this sense, the fixed points $\mathbf{\Phi}_\varepsilon$ determined through (76) and (79) exhibit the universal behavior of the inertial range statistics.

To illustrate this phenomenon, first consider the mean-field regime with $\varepsilon < 2$ and the anisotropic scaling laws from (9) given by

$$\begin{aligned} &\tilde{\alpha}(\lambda) = \lambda^{\tilde{\theta}}, \qquad \tilde{\theta} < 1 \\ &\tilde{\beta}(\lambda) = \lambda^{2\tilde{\theta}} \\ &\tilde{A}(\lambda) = \lambda^{1+\tilde{\theta}} \end{aligned} \tag{81}$$

In this case, the functions $\tilde{\Phi}^\lambda_{2N} = (R^{\tilde{A}}\mathbf{\Phi})_{2N}$ satisfy the Fokker–Planck equations in (73) with coefficients computed through (81). With (75) and (52) for $\varepsilon < 2$, it is a simple matter for the reader to verify that as $\lambda \to 0$,

$\tilde{\mathbf{\Phi}}^{\lambda} \to \tilde{\mathbf{\Phi}}_{\varepsilon}$, a fixed point with $R^{\tilde{\lambda}}\tilde{\mathbf{\Phi}}_{\varepsilon} = \tilde{\mathbf{\Phi}}_{\varepsilon}$, where $\tilde{\Phi}_{2N,\varepsilon}$ is determined by the degenerate Fokker–Planck equation

$$\frac{\partial \tilde{\Phi}_{2N,\varepsilon}}{\partial t} = [2\kappa + 2V^2 R^{\varepsilon}(0)]\, \Delta_{\mathbf{y}} \tilde{\Phi}_{2N,\varepsilon} \tag{82}$$

with the initial conditions from (77). Similarly, for $\varepsilon < 2$ and the anisotropic scaling laws

$$\begin{aligned} \tilde{\alpha}(\lambda) &= \lambda^{\tilde{\theta}}, \qquad \tilde{\theta} > 1 \\ \tilde{\beta}(\lambda) &= \lambda^2 \\ \tilde{A}(\lambda) &= \lambda^{1+\tilde{\theta}} \end{aligned} \tag{83}$$

it follows by similar reasoning that $\tilde{\mathbf{\Phi}}^{\lambda} \to \tilde{\mathbf{\Phi}}_{\varepsilon}$ with $\tilde{\mathbf{\Phi}}_{\varepsilon}$ a statistical fixed point, where $\tilde{\Phi}_{2N,\varepsilon}$ is determined by the degenerate Fokker–Planck equation

$$\frac{\partial \tilde{\Phi}_{2N,\varepsilon}}{\partial t} = \kappa\, \Delta_{\mathbf{x}} \tilde{\Phi}_{2N,\varepsilon} \tag{84}$$

with the initial conditions from (77). I claim that the statistical behavior in both families of noncanonical fixed points satisfying (81), (82) and (83), (84) respectively can be obtained from the canonical fixed point $\mathbf{\Phi}_{\varepsilon}$ defined through (76), (77) in suitable scaling limits. By applying the scaling transformations

$$x' = \lambda \mathbf{x}, \qquad y' = \tilde{\alpha}(\lambda)\mathbf{y}, \qquad t' = \tilde{\beta}(\lambda) t \tag{85}$$

to the canonical fixed point $\mathbf{\Phi}_{\varepsilon}$ as $\lambda \to 0$ the reader can verify immediately that with $\tilde{\alpha}(\lambda)$, $\tilde{\beta}(\lambda)$ defined by either (81) or (83) one recovers the families of fixed points defined through (82) or (84), respectively, in the limit, $\lambda \to 0$. The canonical statistical fixed point in (76), (77) and defined through the usual diffusive scaling laws with $\alpha(\lambda) = \lambda$ and $\beta(\lambda) = \lambda^2$ has an obvious interpretation as an intermediate asymptotic state[11] compared with the two families of noncanonical fixed points. The scaling laws in (81) apply at shorter renormalized times as regards the x-diffusion scaling in the canonical fixed point, while the scaling laws in (83) apply at coarser y length scales as regards the y diffusion in the canonical fixed point; thus, the canonical statistical fixed point arises between a balance of these two competing effects as an intermediate asymptotic state which contains both types of statistical behavior in these respective scaling regions in suitable rescaled limits.

There is similar behavior as described in the above paragraph in the anomalous regime with $2 < \varepsilon < 4$. With the scaling laws analogous to (81) given by

$$\begin{aligned} \tilde{\alpha}(\lambda) &= \lambda^{\tilde{\theta}}, \qquad \tilde{\theta} < \frac{\varepsilon}{2} \\ \tilde{\beta}(\lambda) &= \lambda^{2\tilde{\theta}+2-\varepsilon} \\ \tilde{A}(\lambda) &= \lambda^{1+\tilde{\theta}} \end{aligned} \tag{86}$$

it follows from (51), (52), and (73) that there is a family of statistical fixed points $\tilde{\Phi}_\varepsilon$ in the limit $\lambda \to 0$ with $R^{\tilde{A}}\tilde{\Phi}_\varepsilon = \tilde{\Phi}_\varepsilon$ and $\tilde{\Phi}_{2N,\varepsilon}$ satisfies the degenerate version of the Fokker–Planck equation in (77) with $\kappa = 0$; these fixed points correspond to shorter renormalized times as regards the x-diffusion scaling in the canonical fixed point $\mathbf{\Phi}_\varepsilon$ defined through (76), (77), and the scaling laws in (57). With the scaling laws analogous to (83) with

$$\begin{aligned} \tilde{\alpha}(\lambda) &= \lambda^{\tilde{\theta}}, \qquad \tilde{\theta} > \frac{\varepsilon}{2} \\ \tilde{\beta}(\lambda) &= \lambda^{2} \\ \tilde{A}(\lambda) &= \lambda^{1+\tilde{\theta}} \end{aligned} \tag{87}$$

a second family of noncanonical statistical fixed points arises as $\lambda \to 0$; here the components of the fixed point vector $\tilde{\mathbf{\Phi}}_\varepsilon$ satisfy the degenerate Fokker–Planck equation in (84). These fixed points $\tilde{\mathbf{\Phi}}^\varepsilon$ are associated with the scaling laws in (87), which apply at coarser y length scales as regards the y diffusion in the scaling laws from (60) for the canonical fixed points. As for the mean-field regime, in the anomalous region with $2 < \varepsilon < 4$, these statistical fixed points $\tilde{\mathbf{\Phi}}_\varepsilon$ can be recovered in the limit $\lambda \to 0$ from the canonical fixed point $\mathbf{\Phi}_\varepsilon$ defined through (79), (77), and (60) by applying the scaling transformations in (85) based on either (86) or (87) to $\mathbf{\Phi}_\varepsilon$; thus, the canonical fixed point $\mathbf{\Phi}_\varepsilon$ is an intermediate asymptotic state in the anomalous regime. It is worth mentioning here that the changes in scaling behavior from $\varepsilon < 2$ to $\varepsilon > 2$ through the "phase transition" provide another explicit connection between the renormalization theory for the model and intermediate asymptotics in the Fokker–Planck PDEs which arises in the crossover to anomalous behavior for $\varepsilon > 2$.[11,12]

5. CONCLUDING DISCUSSION

In this paper, the inertial range renormalization theory has been developed in detail for a simple model for turbulent diffusion with statistical velocity fields having long-range correlations in space and white noise correlations in time. To the author's knowledge, this is the first example of turbulent diffusion with the velocity statistics having some essential features of developed turbulence where the inertial range renormalization theory has been developed in full rigorous detail without any ad hoc approximations.

First, renormalization in the high-Reynolds-number limit was developed in Section 2 through an exact quantum mechanical analogy for (2) combined with some simple analytic facts and algebraic manipulation. In Sections 3 and 4, the scale-invariant inertial range renormalized fixed point theory for the vector of higher-order statistics **P** was developed in detail for Gaussian random initial data. There is phase transition in both the canonical scaling laws and the structure of the renormalized fixed point as the spectral parameter ε, measuring the strength of long-range correlations in the velocity field, crosses over from the mean-field regime with $\varepsilon < 2$ to the anomalous regime with $\varepsilon > 2$. The quantum analogy was exploited in Section 3 to yield features of anomalous turbulent decay in the inertial range; the interesting features of noncanonical fixed points and explicit connections with intermediate asymptotics for the renormalization theory have also been developed in detail in Section 4. The model analyzed here is a model for turbulence much like the spherical model[10] in critical phenomena, where analytic formulas rather than complex diagrammatic perturbation theory can be used in renormalization to yield scaling laws, statistical fixed points, etc.

It is interesting to compare and contrast the results obtained here with those developed by Avellaneda and the author in earlier work.[6-8] The principal goal in refs. 6 and 7 was to study eddy diffusivity theory, a problem of great practical importance, for the model in (2) with the large-scale isotropic deterministic initial data

$$T\,|_{t=0} = T_0(\delta x, \delta y), \qquad \delta \ll 1 \tag{88}$$

Thus, the initial data varies on the integral scale [see (5) above]. The main problem studied in refs. 6 and 7 involved the effect of the velocity statistics on scaling theories and effective equations for the large-scale, long-time average, $\langle T(x/\delta, y/\delta, t/\rho^2(\delta))\rangle$, involving the mean statistics. Such simplified equations are equations for eddy diffusivity since they incorporate the coupling of the statistical regime of velocity scales to the integral scales for the diffusing scalar through simpler deterministic equations for the mean, $\langle T(x/\delta, y/\delta, t/\rho^2(\delta))\rangle$, varying only on the large scales. The theory in refs. 6 and 7 involves velocity statistics depending on two parameters, ε and z; as explained in ref. 8, the velocity statistics utilized in this paper correspond to an extreme limiting case from refs. 6 and 7 with $z=0$ and ε varying. Both the inertial range scaling theory developed here and the scaling theory for eddy diffusivity in this special case with $z=0$ exhibit a phase transition with anomalous behavior for ε with $2<\varepsilon<4$ and mean-field behavior for ε with $-\infty<\varepsilon<2$. However, in refs. 6 and 7, the *inertial range scaling parameter* λ *from this paper is not taken as independent, but as a consequence of (88) is necessarily linked to the initial data* through the identification $\lambda=\delta$, $\tilde{\alpha}(\lambda)=\delta$, $\tilde{\beta}(\lambda)=\rho^2(\delta)$—the result is that the equations for the mean, $\bar{T}=\lim_{\delta\to 0}\langle T^\delta\rangle$, are not scale invariant[7] in the anomalous regime $2<\varepsilon<4$ and depend on the infrared cutoff $\rho_0(|k|)$. Such results for eddy diffusivity theories for a diffusing scalar are in agreement with the conventional wisdom of the turbulence community,[2,3] where it was suggested long ago[19] that the mean statistics of the diffusing scalar are dominated by the velocity scales with the most energy and thus depend on velocities near the integral scale for $2<\varepsilon<4$ as reflected through the cutoff $\rho_0(|k|)$. In ref. 7, the *second-order statistics* are also studied in the limit with the inertial range parameter λ linked with δ via the *same scaling law as the mean statistics,* $\lambda=\delta$, $\tilde{\alpha}(\lambda)=\delta$, $\tilde{\beta}(\lambda)=\rho^2(\delta)$. This approach yields the behavior of the second-order statistics at scales in the inertial range necessarily very close to the integral scales and includes interesting qualitative behavior in the model for physical laws[7] such as the Richardson 4/3 law, etc.; however, the equations and scaling laws in ref. 7 for the second-order statistics do not apply throughout the inertial range and do not correspond to the canonical fixed point for the second-order statistics P_2; these results are developed here in Sections 3 and 4.

With the results from refs. 6 and 7 and other recent work[13,18] claiming to study the inertial range renormalization theory in the models from (2) solely through the mean statistics, it is interesting to comment briefly on the behavior of the scaling theory for the mean statistics for the scalar T with the velocity statistics in (3)–(7) and deterministic initial data. It follows from similar manipulations as developed in Section 2 [see Eq. (3.43) in ref. 8] that with

$$\bar{T}^{\bar{\lambda},\delta}=[\lambda\alpha(\lambda)]^{-1}\left\langle T\left(\frac{x}{\lambda},\frac{y}{\tilde{\alpha}(\lambda)},\frac{t}{\tilde{\beta}(\lambda)}\right)\right\rangle \tag{89}$$

the mean statistical quantity $\bar{T}^{\bar{\lambda},\delta}$ satisfies the diffusion equation

$$\begin{aligned}\frac{\partial\bar{T}^{\bar{\lambda},\delta}}{\partial t}&=\frac{\lambda^2}{\tilde{\beta}}\kappa\frac{\partial^2}{\partial x^2}\bar{T}^{\bar{\lambda},\delta}+\frac{\tilde{\alpha}^2}{\tilde{\beta}}[\kappa+R_\delta^\varepsilon(0)]\frac{\partial^2}{\partial y^2}\bar{T}^{\bar{\lambda},\delta}\\ \bar{T}^{\Lambda,\delta}|_{t=0}&=[\lambda\tilde{\alpha}(\lambda)]^{-1}T_0\left(\frac{x}{\lambda},\frac{y}{\tilde{\alpha}(\lambda)}\right)\end{aligned} \tag{90}$$

with $R_\delta^\varepsilon(0)$ given in (4). If, as in Section 2, the high-Reynolds-number limit $\delta \to 0$ is taken in (90), then

$$R_\delta^\varepsilon(0) \to \infty \qquad \text{for} \quad 2 < \varepsilon < 4$$

and the equations for the mean statistics for the scalar as given in (90) do not renormalize to finite quantities unless λ, $\tilde{\alpha}(\lambda)$, $\tilde{\beta}(\lambda)$, and δ are linked as, for example, in refs. 6 and 7. This divergence has lead some authors[13,18] to claim that the problem in (2) is not renormalizable in the inertial range for $2 < \varepsilon < 4$; in that work,[13] scale invariance of equations for the mean statistics is restored by artificial, nonphysical devices, such as time-dependent infrared cutoffs (to quote from ref. 13, "time dependence in the infrared cut-off ... is utilized to achieve consistency between the asymptotic scaling exponents and the scaling behavior of the asymptotic equations"). It is well known and standard in the turbulence community that universal scaling behavior for the diffusing scalar should be determined by second-order statistics[2,3] and not the mean statistics. This was already discussed in 1938 by Taylor.[19] Such a fact has been established in this paper; despite the infrared divergence exhibited in (90) as $\delta \to 0$ for the mean statistics, the entire universal scale-invariant renormalized fixed point for the scalar statistics in the inertial range with random initial data has been developed for the model in (2) in Sections 3 and 4 for the anomalous regime $2 < \varepsilon < 4$. Furthermore, the universal scaling theory for the second-order statistics determines the scaling behavior of all higher-order statistical quantities. It would be very interesting to generalize the results in this paper to the more complicated families of velocity statistics in refs. 6 and 7 for the model in (2).

Finally, it is worth mentioning here that in a suitable sense, the random shear flow models discussed here are not only a qualitative "toy" model, but are strongly related to a quantitative model for renormalization theory for the general transport diffusion problem in (1) with isotropic incompressible velocity fields. Such a link has been developed recently by the author[20] with a rigorous mathematical theory of renormalization and will be published elsewhere.

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